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Analysis of inventory models with budget constraint.

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ANALYSIS OF INVENTORY MODELS
WITH BUDGET CONSTRAINT

by

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I. INTRODUCTION

In today's world, while all systems are becoming more and more sophisticated, the control and maintenance of inventories of these systems is a problem common to all enterprises and military services. In private and commercial concerns the effective control of inventories can result in decreased costs, increased sales and profits and consumer satisfaction. In the military proper management of inventories may contribute to increased availability and readiness, decreased inventory investment and system costs.

For each component of each weapon system two fundamental questions must be answered:

(1) When to replenish the inventory;

(2) How much to buy for the replenishment.

In order to answer these questions, many inventory models have been developed in the past 30 years. See for example, Hadley and Whitin [Ref. 1], Muckstadt [Ref. 2] and Eriksson [Ref. 3]. Most previous work solves a variety of cost minimization problems considering expected values of steady state variable costs associated with shortage cost, ordering cost and storage cost.

Such models may be appropriate for the commercial sector, but are not always appropriate in the military world.

In the commercial sector, the objective function of the inventory model is to maximize profit or minimize the average annual costs. Non-cost oriented objective functions frequently are used in the military inventory systems. For example, attempts are often made to maximize availability or fill rate, or minimize the number of backorders or expected time weighted stockouts, or minimize the probability of a stock-out with a budget constraint.

Obviously costs are important in every inventory model. However, many real-world inventory problems are so complicated, one cannot represent accurately the real situation. Thus, some simplifications and approximations are used when constructing a mathematical model of any real world system. If this is not done, the results obtained by use of the model can easily lead to operating rules which are worse than those currently in use, worse than those which could be derived from simple heuristic intuitive considerations.

Many of the inventory problems are viewed as single period problems. For example, initial provisioning, allowance list determination and the fly-away kit problem are single period problems. These models are perhaps the simplest of the models in which demand is treated as a stochastic variable.

Reasonable objective functions in these models are to maximize performance subject to a constraint on the resources. Typical measures of performance might be availability, time-weighted units short, fill rate, the number of backorders, and mean supply response time.

This thesis considers various single period models which attempt to maximize performance subject to budget constraint.

Chapter II describes the general single period problem and introduces the method used in this thesis of solving those problems.

Chapters III and IV develop the time-weighted units short model and the availability model, and explain the solution procedure. Sample data runs for both models are provided.

Chapter V provides a comparison of models considered in the thesis and discusses some of the properties of each model, and Chapter VI discusses the use of models for purposes of determining the budget required.

Chapter VII summarizes the results of the research and concludes with some suggestions for additional research.

II. THE GENERAL PROBLEM

In this chapter, we consider the general single period model as a process for transforming resources into new distributions of inventory positions over the line items in the inventory.

The essential problems of control in a line item-inventory control system with multiple line items are:

(1) How much resources to commit at a point in time;

(2) How shall these resources be allocated to achieve system objectives.

In a typical continuous review inventory system, we can determine the optimal order quantity (Q) and reorder point (r) for a given item by minimizing the average annual variable costs. But in applying this theory to the real world inventory systems which consist of multiple line items, it is frequently the case that resulting minimum cost solutions are not feasible because of a budget limitation or some other constraint. Thus, in a constrained multi-item inventory system, the typical continuous-review policy is sometimes inappropriate. In the following section we discuss several objective functions to guide the line item inventory control system in determining how to allocate availabile procurement funds at a particular replenishment epoch.

A. GENERAL FORM OF OBJECTIVES WITH CONSTRAINTS

Consider the case in which an administrator, responsible for the replenishment decisions, determines replenishment of stocks of various line items on a periodic basis. Suppose that a fixed amount of procurement budget has been allocated to the replenishment epoch at hand and that a target number of reorder actions has been established as a working constraint for the allocation epoch. The administrator's task is to transform the available resources into replenishment orders for different items.

1. Measure of Effectiveness and Objective Function

Daeschner [Ref. 5] examined the constrained line-item allocation problems. He considered several possible objective functions which can be adapted to the case where unsatisfied demands are backordered and to the case where unsatisfied demands are lost sales. Let $\pi_j > 0$ be the penalty (reward) per unit for item j and let D_j be the demand for item j in a period. Let X_j be the inventory position for item j after ordering in a period. Let $D_j = d_j$. Then the number of sales for item j in the period is given by

$$\begin{array}{lll} d_i & \text{if} & d_j \leq X_j \\ X_j & \text{if} & d_j > X_j \end{array}$$

The expected sales for item j is therefore

$$\sum_{d_j=1}^{X_j} d_j\ p(D_j = d_j) + X_j\ p(D_j > X_j)$$

which is equivalent to

$$E(D_j) - \sum_{d_j=X_j+1}^{\infty} (d_j - X_j) p(D_j = d_j) \ .$$

We assume that the inventory system seeks to minimize the expected penalty incurred, or, equivalently, to maximize the expected penalty avoided. Mathematically, the objective is to maximize

$$Z(\underline{X}) = \sum_{j=1}^{N} \pi_j (E(D_j) - \sum_{d_j=X_j+1}^{\infty} (d_j - X_j) p(D_j = d_j))$$

Several interpretations and uses of the penalty coefficient π_j are possible. Four are illustrated in Table I below. Each reflects a formulation of system objectives which has been adopted or considered by the Navy Supply System. Daeschner [Ref. 5] also considered π_j as a linear combination of various coefficients in his line item allocation model.

There are many other types of objective functions which are currently used in the military. For example,

(1) Minimize units short in a given period

$$Z(\underline{X}) = \sum_{j=1}^{N} \sum_{d_j=X_j+1}^{\infty} (d_j - X_j)\ p(D_j = d_j)$$

TABLE I

INTERPRETATIONS AND USES OF π_j

Penalty Coefficients	Objective
$\pi_j = c_j$	Maximize expected sales from stock.
$\pi_j = 1/\mu_j$	Maximize the expected requisitions filled (μ_j = average quantity of item j demanded per requisition).
$\pi_j = 1$	Maximize the expected number of units issued from stock.
$\pi_j = LT_j + TMNIS - TMISS$	Maximize expected customer waiting time per unit avoided by issue from stock, where LT_j is the lead time for item j, TMNIS is the calendar time anticipated to process a request and TMISS is the time to affect issue from stock of a demand, available item.

(2) Minimize time-weighted units short

$$Z(\underline{X}) = \sum_{j=1}^{N} TWUS_i(X_i) \cdot E_i$$

where:

$TWUS(X_i)$ = time weighted units short

E_i = essentiality

(3) Maximize system availability

$$Z(\underline{X}) = \prod_{i=1}^{N} A_i(X_i)$$

where:

A_i = Availability for each item.

The objective functions (2) and (3) will be explained in Chapters III and IV.

2. <u>The Line-Item Allocation Model and Solution Procedure</u>

In the previous sections, many kinds of objective functions are introduced. If we define them correctly, those objective functions can be solved by various techniques. It is evident that an actual inventory system with limited resources might be unable to carry out a prescribed inventory policy if either the amount of procurement funds available or the number of replenishment actions exceed the available resources. The problem is made more complicated by the fact that the objective functions are "non-linear" and the requirement that the X_j's must be integers. The problem is stated mathematically as

$$\max \quad Z(\underline{s})$$

$$\text{(A1)} \quad \text{s.t.} \quad \sum_{j=1}^{N} c_j s_j \leq B$$

$$\sum_{j=1}^{N} H(s_j) \leq R$$

$\underline{s} = (s_1, s_2, \ldots, N)$: integers

$c_j =$ the unit price of item j

$s_j =$ the number of buys of item j

$H(s_j) = 1$ if $s_j > 0$

$= 0$ otherwise

$B =$ the procurement budget limit at the realloca-tion epoch

$R =$ the maximum number of individual procurement activities allowed in the present allocation.

To solve the problem (A1), the generalized Lagrange multiplier (GLM) method of Everett [Ref. 4] can be used. Using this method, the problem can be reexpressed as

$$\text{(A2)} \qquad \max_{\underline{s}} \; L(\underline{S},\underline{\lambda}) = Z(\underline{s}) - \lambda_1\left(\left(\sum_{j=1}^{N} c_j S_j\right) - B\right)$$

$$- \lambda_2\left(\left(\sum_{j=1}^{N} H(s_j)\right) - R\right)$$

$\underline{S} \in s$ and $\lambda_1, \lambda_2 \geq 0$ with optimal solution $\underline{S}^*(\underline{\lambda})$.

Problem (A2) is the Lagrangian problem associated with (A1). Using Everett's theorem, one can determine a bound on the optimal solution, $Z(\underline{s}^*)$ to be

$$Z(\underline{s}^*) \leq Z(\underline{s}^*(\underline{\lambda})) - \lambda_1(B(\underline{\lambda}) - B) - \lambda_2(R(\underline{\lambda}) - R)$$

where

$$B(\underline{\lambda}) = \sum_{j=1}^{N} c_j S_j^*(\underline{\lambda})$$

$$R(\underline{\lambda}) = \sum_{j=1}^{N} H(S_j^*(\underline{\lambda}))$$

and $\underline{s}^*(\lambda)$ is the optimal solution vector for a given pair (λ_1, λ_2). In solving problem (A2), we can separate the N-variable optimization problem into N one-variable problems. Choosing trial values of λ_1 and λ_2 we maximize

$$\text{(A3)} \qquad L_j(s_j, \underline{\lambda}) = z_j(s_j) - \lambda_1 c_j s_j - \lambda_2 H(s_j) \ .$$

When considering the integer nature of decision variables, the optimal solutions for (A3) are determined by finding the values s_j^* such that

$$L_j(s_j^*+1, \underline{\lambda}) - L_j(s_j^*, \underline{\lambda}) \leq 0 \quad \text{and} \quad L_j(s_j^*, \underline{\lambda}) - L_j(s_j^*-1, \underline{\lambda}) > 0.$$

Thus s_j^* is the smallest value such that

$$\Delta L_j(s_j, \underline{\lambda}) = L_j(s_j+1, \underline{\lambda}) - L_j(s_j, \underline{\lambda}) \leq 0$$

In order to get an optimal solution, Daeshner [Ref. 5] used an interactive computer program, which evaluates the current

optimal solutions with λ_1 and λ_2. Each time, the user can select a pair λ_1, λ_2 and objective function type to be considered. Then the user is provided with output which indicates the budget consumed, the number of stock replenishments generated, the achieved objective function value and a maximum attainable value for the objective function.

After examining the output, the user can modify the input parameters and continue or terminate the run. Decreasing the non-negative multiplier values tends to use more of the corresponding resources, increasing the values used, less. When the replenishment actions generated by a pair of values (λ_1, λ_2) exactly consume the available resources, B and R, the solution is optimal. Frequently exact equality may be impossible because of integer nature of the problem. Thus the solution obtained may not be optimal, but the difference is not likely to be significant.

B. AUTOMATING SEARCH ON THE LAGRANGE MULTIPLIER

The interactive search method cannot guarantee an optimal solution and it requires trial and error to get the approximate optimal solution. Consider the case in which there is only a single constraint, with the same type of objective functions. The mathematical program is then

$$\text{(B1)} \qquad \max_{\underline{s}} Z(\underline{s}) \qquad \text{s.t.} \quad \sum_{j=1}^{N} c_i s_i \leq B$$

$\underline{s} = (s_1, s_2, \ldots, s_N) =$ integer number of buys

$B =$ budget limit

$c_i =$ price of item i.

We can rewrite the above equation using a Lagrange multiplier, as:

$$(1) \qquad L(\underline{s}, \theta) = Z(\underline{s}) - \theta\left[\sum_{j=1}^{N} c_i s_i - B\right]$$

Then separate the equation.

$$(2) \qquad L(s_1, s_2, \ldots, s_N) = \sum_{j=1}^{N} (Z(s_i) - \theta c_i s_i) + \theta B$$

Equation (2) can be maximized by maximizing each sub-objective function. If $Z(s_i)$ is differentiable with respect to each s_i, the optimal solution is obtained by

$$\frac{\partial L}{\partial s_i} = \frac{dZ(s_i)}{ds_i} - \theta c_i, \qquad i = 1, 2, \ldots, N$$

Thus set $\frac{\partial L}{\partial s_i} = 0$ and get

$$\theta = \frac{dZ_i(s_i)}{ds_i} / C_i$$

where θ is such that $\sum_{j=1}^{N} c_i s_i = B$. Everett [Ref. 4] shows that θ can also be interpreted as a shadow price for the objective function: i.e., $\theta = \partial Z/\partial B$. Due to the integer

nature of s_i, it is often impossible to get an exact optimal solution. Difference equations must be used because the region in which the solution is desired consists of a set of discrete points. Therefore, let

$$(3) \qquad \Delta L_i(s_i,\theta) = L_i(s_i,\theta) - L_i(s_i-1,\theta)$$

$$= Z_i(s_i) - \theta c_i s_i - Z_i(s_i-1) + \theta c_i(s_i-1)$$

$$= \Delta Z_i(s_i) - c_i\theta$$

We know that Equation (3) is a concave function at the point $\underline{s} \geq 0$. The optimal solution must satisfy $\Delta L_i(s_i,\theta) \geq 0$ and $\Delta L_i(s_i+1,\theta) < 0$. Thus the optimal solutions are given by finding the largest s_i's such that

$$\Delta L_i(s_i,\theta) \geq 0$$

or equivalently

$$(4) \qquad \Delta Z_i(s_i) - c_i\theta \geq 0 \qquad i = 1,\ldots,N.$$

The Lagrangian multiplier θ can be found by the following search algorithm.

STEP 1. Find an initial upper bound θ_u. Let all s_i be assigned zero at the beginning and find the change of objective function per unit dollar as a result of increasing to one unit.

$$\theta_1 = \frac{\Delta Z_1(1)}{c_1}$$

$$\theta_2 = \frac{\Delta Z_2(1)}{c_2}$$

$$\vdots$$

$$\theta_n = \frac{\Delta Z_n(1)}{c_N}$$

where $\Delta Z_i(1) = Z_i(1) - Z_i(0)$.

Because of decreasing marginal returns or objective function values and because of the interpretation of θ, an upper bound on θ is given by: $\theta_u = \max[\theta_1, \theta_2, \ldots, \theta_n]$.

STEP 2. The initial θ_o will be

$$\theta_o = \frac{\theta_L + \theta_u}{2}$$

where $\theta_L = 0$.

Find for each i, the largest s_i so that

$$\frac{\Delta Z_i(s_i)}{c_i} \geq \theta_o$$

and evaluate the objective function and the budget required.

STEP 3. If the budget used is greater than the given budget, let

$$\theta_1 = \frac{(\theta_o + \theta_u)}{2}$$

otherwise

$$\theta_1 = \frac{\theta_o + \theta_L}{2}$$

Each time update the S vector, the objective function values, the upper bound of the objective, and the amount of budget consumed.

STEP 4. Stopping rule.

Stop when the used budget is equal to the given budget or the difference between the current upper bound and the objective function value is less than some limit (ε). Otherwise go to step 2, and continue until the above conditions are satisfied.

A FORTRAN program for this algorithm is given in Appendix B.

C. MARGINAL ANALYSIS PROCESS

The theory of marginal analysis has been used in many inventory models when resource constraints are active. In an economic sense, $\Delta Z_i(s_i)/c_i$ can be interpreted as the marginal increase in the objective function per dollar spent

achieved by adding one more unit of stock. It is reasonable for an inventory controller who has a scarce resource such as a procurement budget to buy an item which gives the maximum benefit per dollar spent.

By using a simple computerized algorithm, the line item allocation problem can be solved easily. The first step is to set all $s_i = 0$ and compute

$$\max_i \left[\frac{\Delta Z_1(s_1+1)}{c_1}, \frac{\Delta Z_2(s_2+1)}{c_2}, \ldots, \frac{\Delta Z_n(s_n+1)}{c_n}\right]$$

If the maximum is taken on for item j, set $s_j = 1$ and deduct the unit price for unit j from the budget. The second step is then to recompute ΔZ_j and then find

$$\max\{\max_{i \neq j}\{\frac{\Delta Z_i(s_i)}{c_i}\}, \frac{\Delta Z_j(s_j)}{c_j}\} .$$

The next unit is assigned to the index j where the maximum is taken on, etc. This is continued until adding an additional unit exceeds the budget constraint. It should be noted, however, that the method described does not insure optimality [Ref. 1]. Specifically the method may stop too soon. If the item i selected from the marginal analysis has a c_i value greater than the remaining budget, the procedure terminates even though some other item j may have a c_j value less than the remaining budget. An obvious improvement in this area could be the inclusion of a subroutine that would select

from the remaining items the best one from those having c_j's smaller than the remaining budget. A FORTRAN program for performing this marginal analysis is provided in Appendix C.

D. SAMPLE DATA RUNS OF UNITS SHORT MODEL

A weapon system consists of 10 components. The system manager wants to minimize the number of units short by supplying spare parts to support the weapon system. Suppose that the demand rate, lead time, price and essentiality code for each item i are known. The objective function can be expressed by

$$\text{minimize } Z(\underline{s}) = \sum_{i=1}^{10} \sum_{d_i=s_i+1}^{\infty} (d_i - s_i)\, p(D_i = d_i)\, E_i \tag{C1}$$

$$\text{subject to} \quad \sum_{i=1}^{10} s_i c_i \le B$$

where:

E_i = essentiality code

B = budget limit

$$p(D_i = d_i) = \frac{e^{-\lambda_i T_i} (\lambda_i T_i)^{d_j}}{(d_i)!}$$

The approximate solution of (C1) can be obtained by the marginal analysis method. Table II shows the computational results for this system with known input data.

TABLE II

THE RESULT OF UNITS SHORT MODEL

No.	λ_i	Lead Time	Price ($)	Essen.	Allocation	Units Short
1	1.0	1.0	10.2	1.0	5.0	0.0007
2	0.1	1.0	20.0	1.0	1.0	0.0048
3	3.0	1.0	100.0	1.0	2.0	1.2489
4	25.0	1.0	2.0	3.0	42.0	0.0013
5	1.0	1.0	5.0	1.0	5.0	0.0007
6	0.5	1.0	5.0	3.0	4.0	0.0002
7	10.0	1.0	1.0	1.0	21.0	0.0012
8	5.0	1.0	100.0	1.0	4.0	1.4368
9	1.0	1.0	50.0	1.0	3.0	0.0233
10	2.0	1.0	100.0	1.0	2.0	0.5413

Table II shows several properties of the units short model. First of all, more than one unit short in a year occurs in the high cost items (items 3 and 8). Second, low demands and low price items are allocated enough. Items 2, 5 and 6 are allocated more than five times their mean demand. Also this model tends to stock more of the high demands and low price items.

Finally, the essentiality weights cause greater allocations to be provided to those items with high essentiality than would be provided with equal weights.

Other results include:

Total objective value	3.26
Shadow price	0.001899
Budget limit	\$1170
Budget left	\$0.0

The shadow price is the last maximum value of $\frac{\Delta Z(s_i)E_i}{c_i}$. It can be interpreted approximately as the amount of decrease in the objective function achieved by adding one more dollar.

III. TIME WEIGHTED UNITS SHORT MODEL

A. DESCRIPTION OF MODEL

In the previous chapter we have discussed various objective functions and solution methods for the single period inventory problem. In the military services, many measures of effectiveness have been used to indicate system performance. Among these measures are fill rate, availability, mean supply response time, the number of stockouts, and time-weighted units short. In this chapter we consider a model which minimizes time-weighted-units-short (TWUS).

Suppose that a weapon system consists of n components and the objective is to allocate a given budget for spare parts so as to minimize time-weighted-units short for the entire system. Assume that

(1) procurement lead time and repair lead time are known constants.

(2) demands for each installed unit have a known distribution.

(3) the total amount of procurement budget available to spend on all components is fixed.

(4) the objective is to minimize essentiality weighted TWUS. Mathematically, the model can be written as

$$\min_{\underline{s}} \sum_{i=1}^{n} TWUS_i(s_i)E_i \cdot \frac{1}{SLT}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_i s_i \leq B$$

where:

$TWUS_i(s_i)$ = time weighted units short when there are s_i units for item i

SLT = total sum of lead time demand $\left(\sum_{i=1}^{n} \lambda_i T_i\right)$

E_i = essentiality code for item i

B = budget limit in a given period

C_i = price of each item.

In the above problem, if the TWUS is properly defined, this model will be solved easily by using the methods explained in Chapter II.

B. POISSON DEMAND CASE

We shall now determine an exact expression for the $TWUS_i(s_i)$ for the case in which demands are Poisson distributed. Let the mean rate of demand be λ_i units per year and the lead time be a constant T_i. In addition to treating the demand variable as being discrete, the number of buys s_i also will be treated as a discrete variable. Thus if D_i is the lead time demand item j:

$$p(D_i=s_i) = \frac{e^{-\lambda_i T_i}(\lambda_i T_i)^{s_i}}{(s_i)!} \tag{1}$$

$$= p(s_i;\lambda_i T_i)$$

Let

$$\bar{p}(s_i) = \text{prob}(D_i \geq s_i) = \sum_{d_i=s_i}^{\infty} p(d_i;\lambda_i T_i) \tag{2}$$

If there are s_i units of stock for item i, Richards and McMasters [Ref. 8] show that the expected time-weighted units short in $(0,T_i)$ is given by

$$E[TWUS_i(s_i)] = \frac{T_i}{2}\{\bar{p}(s_i+1)[\lambda_i T_i - 2s_i + \frac{s_i(s_i+1)}{\lambda_i T_i}]$$

$$+ p(s_i;\lambda_i T_i)(\lambda_i T_i - s_i)\} \tag{3}$$

For those cases where the expected lead time demand is large, the Poisson probabilities in (3) can be approximated by a normal distribution with mean $\lambda_i T_i$ and variance $\sigma_i^2 = \lambda_i T_i$. Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

be the standard normal probability density function and let $\Phi(x) = \int_x^{\infty} \phi(u)du$ be the complementary cumulative distribution

function for the standard normal. Then expression (3) can be rewritten in terms of the normal probability function as follows:

$$E[TWUS_i(s_i)] = \frac{T_i}{2}\{\Phi(\frac{s_i+1-\lambda_i T_i}{\sigma_i})[\lambda_i T_i - 2s_i + \frac{s_i(s_i+1)}{\lambda_i T_i}]$$

$$+ \frac{1}{\sigma_i}\phi(\frac{s_i-\lambda_i T_i}{\sigma_i})(\lambda_i T_i - s_i)\} \tag{4}$$

This expression should be used in those cases in which $\lambda_i T_i$ is large. We have developed the expression for the expected time-weighted units short in a period of length T_i when there are s_i units of stock for item i.

In the next section we write the expression for the total essentiality-weighted time weighted units short over all items, and we provide a solution procedure for allocating the given budget optimally.

C. SOLUTION PROCEDURE

The mathematical program for the time-weighted-units short problem is:

$$\text{(C1)} \quad \min Z(\underline{s}) = \frac{1}{SLT}\sum_{i=1}^{N}\frac{E_i T_i}{2}\{\overline{p}(s_i+1)[\lambda_i T_i - 2s_i + \frac{s_i(s_i+1)}{\lambda_i T_i}]$$

$$+ p(s_i;\lambda_i T_i)(\lambda_i T_i - s_i)\}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} c_i s_i \le B$$

To solve this problem we can use the Lagrangian multiplier technique. Let

$$L(s_1, s_2, \ldots, s_n; \theta) = \sum_{i=1}^{N} Z_i(s_i) + \theta(B - \sum_{i=1}^{N} c_i s_i) \qquad (5)$$

Here Equation (5) is separable in the items, and minimization of the total objective function is accomplished by minimizing the individual functions $Z_i(s_i)$ subject to budget constraints. Consider a single item i. Let

$$\Delta L_i(s_i) = L_i(s_i - 1) - L(s_i)$$

$$= Z_i(s_i - 1) - Z_i(s_i) + \theta c_i s_i - \theta c_i(s_i - 1)$$

$$= \Delta Z_i(s_i) + \theta c_i \qquad (6)$$

where

$$\Delta Z_i(s_i) = E_i[TWUS_i(s_i - 1) - TWUS_i(s_i)] \qquad (7)$$

As shown earlier

$$TWUS(s-1) = \frac{T}{2}\{\overline{P}(s)[\lambda T - 2(s-1) + \frac{s(s-1)}{\lambda T}] + p(s-1;\lambda T)(\lambda T - s + 1)\}$$

$$= \frac{T}{2}\{\overline{P}(s)[\lambda T - 2s + \frac{s^2 - s}{\lambda T} + 2] + \frac{s}{\lambda T}p(s;\lambda T)(\lambda T - s + 1)\}$$

$$TWUS(s) = \frac{T}{2}\{\overline{P}(s+1)[\lambda T - 2s + \frac{s(s+1)}{\lambda T} + p(s;\lambda T)(\lambda T - s)\}$$

$$= \frac{T}{2}\{\overline{P}(s)[\lambda T - 2s + \frac{s(s+1)}{\lambda T}] - p(s;\lambda T)(\lambda T - 2s + \frac{s(s+1)}{\lambda T})$$

$$+ p(s;\lambda T)(\lambda T - s)\}$$

so that

$$TWUS(s-1) - TWUS(s)$$

$$= \frac{T}{2}\{\overline{P}(s)[2 + \frac{s^2 - s - s^2 - s}{\lambda T}] + p(s;\lambda T)[s - \frac{s^2}{\lambda T} + \frac{s}{\lambda T} + \lambda T - 2s$$

$$+ \frac{s^2 + s}{\lambda T} - \lambda T + s]\}$$

$$= \frac{T}{2}[\overline{P}(s)[2 - \frac{2s}{\lambda T}] + \frac{2s}{\lambda T} p(s;\lambda T)]$$

$$= \overline{P}(s)[T - \frac{s}{\lambda}) + \frac{s}{\lambda} p(s;\lambda T) \tag{8}$$

Substitute Equations (7) and (8) into (6). Then

$$\Delta L_i(s_i) = E_i[(T_i - \frac{s_i}{\lambda_i})\overline{P}(s_i;\lambda_i T_i) + \frac{s_i}{\lambda_i} p(s_i;\lambda_i T_i)]$$

$$+ \theta c_i \tag{9}$$

The optimum solution s_i^* is the largest s_i such that

$$\Delta L_i(s_i) \geq 0$$

or equivalently,

$$\frac{E_i(Z_i(s_i-1)-Z_i(s_i)}{c_i} = \frac{E_i}{c_i}[(T_i - \frac{s_i}{\lambda_i})\overline{P}(s_i;\lambda_i T_i)$$

$$+ \frac{s_i}{\lambda_i} p(s_i)] \geq -\theta$$

The basic algorithm for solving this problem was explained in the previous chapter. A computer program for searching for θ is provided in Appendix B.

D. SAMPLE DATA RUNS

Consider a weapon system which consists of 10 components. Suppose that the demand for each component is Poisson distributed with parameter λ_i and the lead time is known constant T_i. Let the budget available for procurement be $19224. Table III shows the optimal allocations provided by the TWUS model.

The allocation given when the demand distribution is approximated by the normal distribution is also provided in Table III for comparison. (For comparability, the variance for the normal distribution is taken to be the same as the mean). Comparing the results, we observe that the normal case buys more of the high demands low cost items. There is a small difference in the allocation for items 8 and 9 which are more expensive than the others. For demand rates less than 10, the usefulness of the approximation is questionable.

TABLE III

OPTIMAL ALLOCATION FOR TWUS MODEL

Item	λ_i	T_i	c_i ($)	E_i	Poisson	Normal
1	10	1	10	1	17	20
2	100	1	20	1	113	120
3	15	1	80	1	19	21
4	20	1	2	1	32	36
5	50	1	5	1	65	71
6	80	1	30	1	90	96
7	20	1	1	1	35	37
8	15	1	200	1	17	15
9	75	1	100	1	77	74
10	10	1	75	1	14	16

The resulting values of the objective function for the optimal solutions are:

	Poisson Case	Normal Case
$Z(\underline{s}^*)$	0.00094	0.0015
Shadow price (θ^*)	0.00015	0.00055
Budget limit	$19224	$19224
Budget left	0	0

The objective function value for the Poisson demand case is less than the normal demand case. The main reason for the difference is due to items 8 and 9.

IV. PSEUDO AVAILABILITY MODEL

A. DESCRIPTION OF MODEL

In the previous chapter, the TWUS objective function was introduced as a means for allocating a limited budget. Operational availability is a widely stated measure of the operational readiness of military forces and weapon systems. Thus, it is appropriate to consider stockage models with an availability objective as a means of allocating limited resources.

> The most direct and meaningful measure of the influence of peace time operating stocks on readiness is weapon system (or end item) availability. We use the terms availability, end-item availability, and weapon system availability interchangeably to mean the probability that an end item, such as a tank or an aircraft, selected at random, is not waiting for a component to be repaired or shipped to it. [Ref. 6]

Many authors have attempted to determine stockage levels for components by maximizing equipment operational availability, subject to a budget constraint. See, for example, Jee [Ref. 7]. Usually, the availability for component i is defined by the ratio

$$A_i = \frac{MTBF_i}{MTBF_i + MTTR_i + MSRT_i(s_i)}$$

where:

$MTBF_i$ = the Mean Time Between Failure of item i;

$MTTR_i$ = the Mean Time To Repair for item i;

$MSRT_i(s_i)$ = the Mean Supply Response Time for item i.

If the weapon system is assumed to consist of the n components all arranged in series, then the system availability is the product of the individual item availabilities. (This assumption means that the system will fail if any of the components fails.) With this assumption, the allocation problem, stated in terms of system availability is:

$$\text{(P1)} \qquad \max_{\underline{s}} \prod_{i=1}^{n} A_i(s_i)$$

$$\text{s.t.} \sum_{i=1}^{n} c_i s_i \leq B \qquad i = 1,2,\ldots,N$$

where:

$A_i(s_i)$ = the availability of item i having s_i units in stock;

B = the budget limit;

C_i = the price for each item i,

In the expression of $A_i(s_i)$, the term $MTBF_i$ is the reciprocal of the failure rate i, $MTTR_i$ is assumed to be independent of the decision variables and the available funds and $MSRT_i(s_i)$ can be expressed in terms of $TWUS(s_i)$ as

$$MSRT_i(s_i) = \frac{1}{\lambda_i T_i} TWUS_i(s_i).$$

Thus, the main determination of availability from the point of view of the supply system is $MSRT_i(s_i)$. Many techniques for solving this model have been developed. In the next section we represent an algorithm for solving the availability model by using the marginal analysis method.

B. SOLUTION PROCEDURE

The model (P1) is not additive in the individual component availabilities but is converted into an additive function by transforming the objective function. Taking the natural log of the objective function, the model can be expressed in the following way.

$$\text{(P2)} \qquad \max_{\underline{s}} \quad \sum_{i=1}^{n} \ell n \ A_i(s_i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_i s_i \le B \qquad i = 1,2,\ldots,N$$

Now the model (P2) is separable for all i and maximization of (P2) yields the same solution as maximization of (P1). The marginal analysis method selects an item which gives at each step the greatest increase in $\log(A_i(s_i))$ per dollar spent.

STEP 1. Start with zero units for all items.

STEP 2. Compute the increase in log availability per dollar spent as a result of purchasing one additional unit.

$$\frac{\Delta \ln A_i(s_i)}{c_i} = \frac{1}{c_i}[\ln A_i(s_i) - \ln A_i(s_i)]$$

where $i = 1,2,\ldots,N$

$$\ln A_i(s_i) = \ln\left[\frac{MTBF_i}{MTBF_i + MTTR_i + MSRT_i(s_i)}\right]$$

STEP 3. Select that item i corresponding to the maximum ratio.

$$\max_{\text{all } s_i} \left[\frac{\Delta\ln A_1(s_1)}{c_1}, \frac{\Delta\ln A_2(s_2)}{c_2}, \ldots, \frac{\Delta\ln A_n(s_n)}{c_n}\right]$$

STEP 4. Increase the number of units stocked for the item selected at step 3 by one additional unit if the unit price is less than the amount of budget remaining.

STEP 5. Update the S vector, the MSRT(s) expression and decrement the available budget. If the remaining budget is greater than the cost of the cheapest item, Go to Step 3. Otherwise, Stop.

In the following section, we will illustrate this procedure with a sample system. The computer program is provided in Appendix C.

C. A NUMERIC EXAMPLE FOR THE MSRT MODEL AND THE AVAILABILITY MODEL

In the expression for availability, the MTBF and MTTR terms are not functions of the number of spare parts. Therefore it is commonly believed that maximization of system availability is equivalent to minimization of mean supply response time. However, this is not the case, as shown below.

Suppose a weapon system consists of three components and the demands are Poisson distributed with parameters λ_1, λ_2 and λ_3, respectively. The lead time is a known constant and the components have essentiality codes E_i. The unit price and MTTR are known and the budget is limited to 20 dollars. This information is summarized in Table IV.

TABLE IV

INPUT DATA FOR EXAMPLE

ITEM	λ_i	c_i	$MTTR_i$	E_i	T_i
1	1	5	0.0274	1	1.0
2	0.1	5	0.0027	3	1.0
3	10	1	0.0054	1	1.0

To solve MSRT minimization problems, we first determine the MSRT's for all possible cases. These values are provided in Table V.

TABLE V

MSRT DATA FOR ALL FEASIBLE SOLUTIONS S_i

$MSRT(s_i)$	ITEM 1	ITEM 2	ITEM 3
MSRT(0)	0.9482	0.9837	0.6
MSRT(1)	0.3161	0.1967	0.5
MSRT(2)	0.0708	0.02	0.4099
MSRT(3)	0.0132	0.00227	0.3298
MSRT(4)	0.0021	0.0002	0.2596
MSRT(5)	0.0003		0.1992
MSRT(6)			0.1485
MSRT(7)			0.1072
MSRT(8)			0.0746
MSRT(9)			0.0500
MSRT(10)			0.0322
MSRT(11)			0.0199

Using the solution procedure described in the previous chapter we determine the optimal solution to be as shown in Table VI.

TABLE VI

THE ALLOCATION OF SPARE PARTS FOR MSRT MODEL

ITEM ALLOCATION	1 $\Delta Z_1(s_1)$	2 $\Delta Z_2(s_2)$	3 $\Delta Z_3(s_3)$	USED BUDGET ($)
(0,0,0)	0.12462	0.47216	0.1	0
(0,1,0)	0.12462	0.1040	0.1	5
(1,1,0)	0.04905	0.1040	0.1	10
(1,2,0)	0.04905	0.01265	0.1	15
(1,2,1)	0.04905	0.01265	0.09	16
(1,2,2)	0.04905	0.01265	0.08012	17
(1,2,3)	0.04905	0.01265	0.07022	18
(1,2,4)	0.04905	0.01265	0.06039	19
(1,2,5)	0.04905	0.01265	0.0507	20

The optimal solution for MSRT model is (1,2,5). Repeating the analysis for the availability objective function we obtain the results provided in Table VII from the marginal analysis procedure.

$$\Delta Z_i(s_i) = \frac{1}{c_i}[Z_i(s_i+1) - Z_i(s_i)]$$

$$= \frac{1}{c_i}[\ln A_i(s_i+1) - \ln A_i(s_i)]$$

TABLE VII

THE ALLOCATION OF SPARE PARTS FOR AVAILABILITY MODEL

ITEM	1	2	3	
ALLOCATION	$\Delta Z_1(s_1)$	$\Delta Z_2(s_2)$	$\Delta Z_3(s_3)$	USED BUDGET
(0,0,0)	0.07712	0.1294	0.1529	0
(0,0,1)	0.07712	0.1294	0.1611	1
(0,0,2)	0.07712	0.1294	0.1689	2
(0,0,3)	0.07712	0.1294	0.1759	3
(0,0,4)	0.07712	0.1294	0.1808	4
(0,0,5)	0.07712	0.1294	0.1821	5
(0,0,6)	0.07712	0.1294	0.1777	6
(0,0,7)	0.07712	0.1294	0.1660	7
(0,0,8)	0.07712	0.1294	0.1469	8
(0,0,9)	0.07712	0.1294	0.1217	9
(0,1,9)	0.07712	0.0447	0.1217	14
(0,1,10)	0.07712	0.0447	0.0938	15
(0,1,11)	0.07712	0.0447	0.06702	16
⋮			⋮	
(0,1,15)				20

The optimal solution for the availability model is (0,1,15). Comparing the results of the two models, we see that the availability model allocates more units to the high demand lower cost items than the MSRT model.

D. SAMPLE DATA RUNS

Suppose that a weapon system consists of 10 components and the demand of each component is Poisson distributed with parameter λ_i, and lead time T_i, mean time to repair $MTTR_i$ are known constants. In order to maximize the availability of spare parts with budget constraint, we can use the modified Availability model (P2) instead of (P1). By using the computer program in Appendix C, this problem can be solved. Table VIII provides the allocations of spare parts in the Availability model when the budget is 1170 dollars.

TABLE VIII

THE ALLOCATION OF SPARE PARTS FOR THE AVAILABILITY MODEL

ITEM	λ_i	T_i	c_i ($)	E_i	$MTTR_i$	ALLOCATION	$A_i(s_i)$
1	1.0	1.0	10.0	1.0	0.0137	4.0	0.986
2	0.1	1.0	20.0	1.0	0.0274	2.0	0.997
3	3.0	1.0	100.0	1.0	0.0137	3.0	0.821
4	25.0	1.0	2.0	3.0	0.0822	37.0	0.327
5	1.0	1.0	5.0	1.0	0.0274	5.0	0.973
6	0.5	1.0	5.0	3.0	0.0027	4.0	0.999
7	10.0	1.0	1.0	1.0	0.0054	21.0	0.949
8	5.0	1.0	100.0	1.0	0.0411	3.0	0.538
9	1.0	1.0	50.0	1.0	0.0082	3.0	0.987
10	2.0	1.0	100.0	1.0	0.1370	2.0	0.697

From the table, one can see that the availability of an item is greatly influenced by the MTTR term (see item 4). The availability for that item never exceeds 0.333 even if the MSRT is zero. We also observe that the availability model tends to stock the high demand low cost items.

The objective function for the optimal solution is given by:

Total obejctive value	0.08999
Shadow price	0.000261
Budget limit	$1170
Budget left	$0.0

A comparison of the above results with the allocation given in Table VIII shows that the total availability is relatively low even though most of the items have high availabilities. Also as mentioned above, when the MTTR data for an item is large relative to the MTBF, a high availability cannot be achieved.

V. COMPARISON OF MODELS

A. ANALYSIS FOR SAME DATA

In this chapter, we continue to consider the allocation of spare parts to maximize the system performance in the different allocation models. In this thesis, we have looked at three models: the units short model, the time-weighted units short model, and the availability model. Since each model attempts to reduce stockouts as much as possible the allocations generated by the models are strongly correlated. This is especially true for the availability model and the MSRT model since availability is a function of MSRT. However, we saw earlier that the allocations from the models are not necessarily the same.

Assume that a weapon system consists of 10 items, the demands are Poisson distributed and $MTTR_i$, c_i, T_i, E_i are known constants and a budget constraint of the weapon system is \$1170. The optimal allocations for the three models are shown in Table IX. As can be seen, the TWUS model is more sensitive to the lead times than are the other two models (see items 5, 6, and 7).

The units short model is more sensitive to the price of the item than are the other two models. For item 9 the units short model bought nothing, but the TWUS model and the availability model allocated 2 and 3 items respectively. All

TABLE IX

THE ALLOCATIONS OF SPARE PARTS FOR THE THREE DIFFERENT MODELS

Item	λ_i (yr)	cost ($)	Ess.	T_i (yr)	MTTR (yr)	Units Short Model	TWUS Model	Avail. Model
1	1.0	10	1	1	0.0137	3	3	4
2	0.1	10	1	1	0.0137	2	1	2
3	15.0	3	1	1	0.0137	24	20	22
4	15.0	3	3	1	0.0274	26	23	24
5	3.0	10	1	0.5	0.0274	4	3	4
6	3.0	5	3	0.5	0.0274	6	4	6
7	10.0	50	1	0.2	0.0054	0	0	3
8	10.0	50	1	1	0.0411	8	7	2
9	2.0	50	1	1	0.0137	0	2	3
10	2.0	100	4	2	0.1370	5	5	5

three models are highly affected by the essentiality code. This is illustrated by a comparison of items 9 and 10. Table X presents the corresponding values of the three objective functions.

TABLE X

THE COMPARISON OF OBJECTIVE VALUES FOR THREE MODELS

OBJ FN / MODEL	UNITS SHORT	TWUS	AVAILABILITY
UNITS SHORT	0.1523	0.0458	0.0483
TWUS	0.1527	0.0353	0.0687
AVAILABILITY	0.1969	0.0775	0.0709

The above table was established by computing each objective function for the allocations determined by the three different procedures. Comparing the results of the three models, the TWUS model seems to do the best job considering all three objective functions. However, no general conclusions can be drawn about the preference of the TWUS model for other situations.

One needs to determine which objective function most closely matches a servicers' feeling about how operational readiness is affected by stockouts and delays in satisfying stockouts.

B. DISCUSSION OF SIMILARITIES

In the budget allocation problem there are many factors which affect the allocation such as demand, lead time, cost, time to repair and essentiality. The three models share similar properties. First of all, as can be seen in the above

example, all models tend to stock the cheap, high demand items in favor of expensive low demands items. This is because of the models attempts to get the biggest benefit per dollar spent. Potential benefit per additional unit increases with an items demand rate. Second, items having high essentiality code are given preference, as is the intent of essentiality assignment schemes. Essentiality weighting is one way to counter the preference given the high demand low cost item observed earlier. It is frequently the case that the most critical items are low demand expensive items. Without the essentiality weighting such items would be neglected by the type of models examined in this thesis.

VI. USE OF THE MODELS FOR BUDGET DETERMINATION

A. EFFECTIVENESS VS. BUDGET

The models that we have discussed have attempted to optimize performance subject to a budget constraint. We have assumed that the budget was given. There are many ways in which budgets are determined. However, budgeting people and inventory managers alike often express the desire to have a methodology that they can use to determine the amount of money that should be provided.

In most cases the amount is determined historically by giving an amount equal to what has been provided in the past for similar systems or perhaps by giving a little more or less based on judgement or financial constraints. There is, however, a strong interest brought about by Congressional pressures to relate resources to readiness. Congress wants to know "how much money is needed to support our weapon systems at a specified level of performance." In this chapter we show how the models developed earlier in this thesis can be used in just this manner.

Specifically, we show how the models that we have developed can be modified easily to determine the minimum budget required to provide a specified level of logistics performance.

The models developed earlier can each be run for a range of budget levels producing for each given budget an allocation

and a predicted overall level of performance. Figure 3 illustrates this for the case in which the performance measure is pseudo-availability. As expected, the curve shows that availability is a non-decreasing function of budget with decreasing marginal returns. This can be done also for the time-weighted units short model or any of the other models discussed in this thesis. In all cases we would obtain a similar display. Performance is a monotonic function of budget with decreasing marginal returns.

Figure 4 displays a similar result for the case in which the performance measure is MSRT. Each point on the curve represents an optimal level of performance for a given budget. For this example displayed, Figure 3, there is a little benefit to be gained by increasing the budget above $2500. However there is a dramatic increase in effectiveness obtained by increasing the budget from $1000 to $2000. This is precisely the sort of information needed to make intelligent budgeting decisions. Of course some decision maker must decide if the increase in effectiveness is worth the additional expenditure.

If a specific level of effectiveness is specified, one can graphically determine the amount of budget required by simply moving horizontally across the graph from the specified level of effectiveness until the curve is intersected and then down to the budget axis.

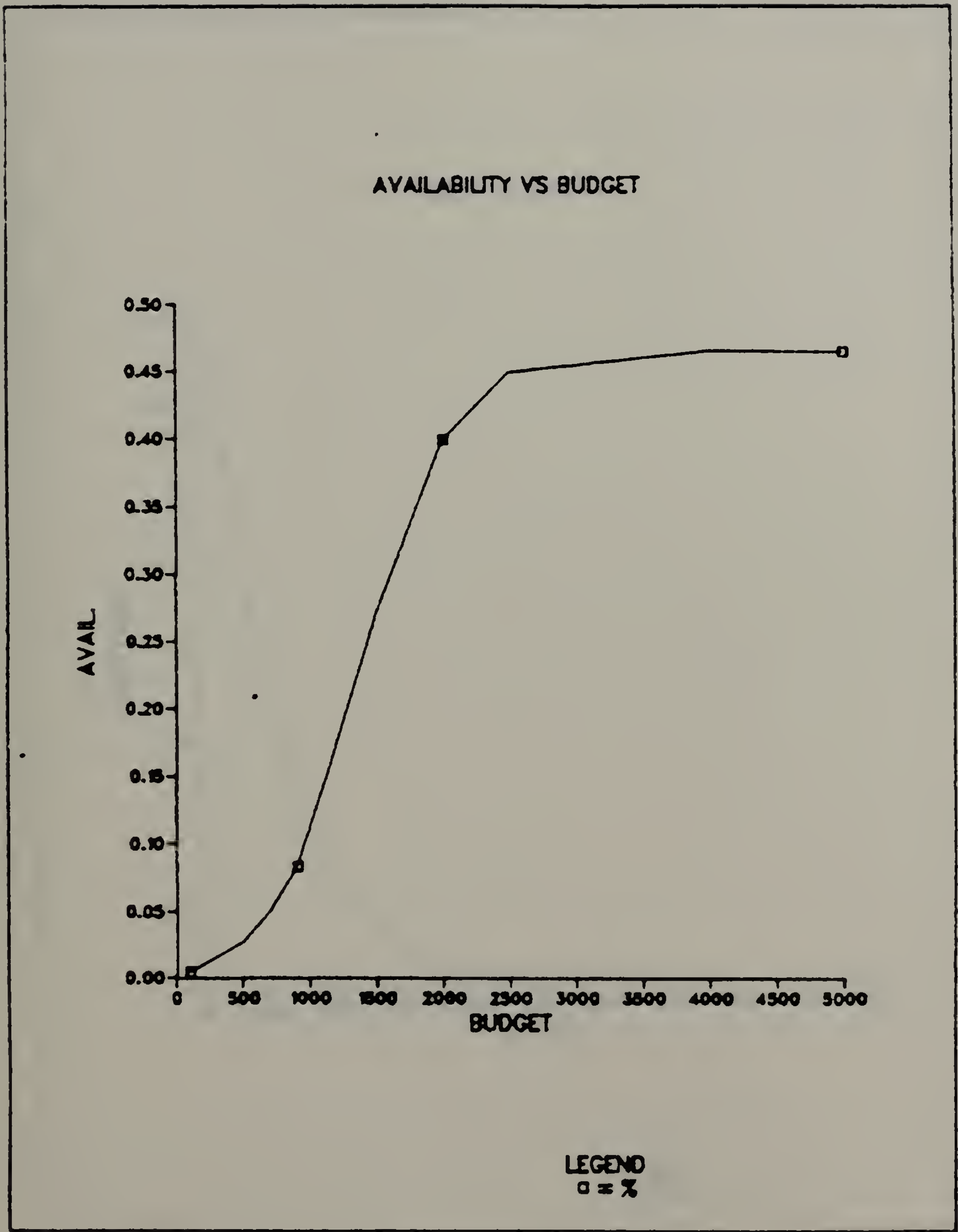

Figure 3. Total Availability Vs. Budget Curve

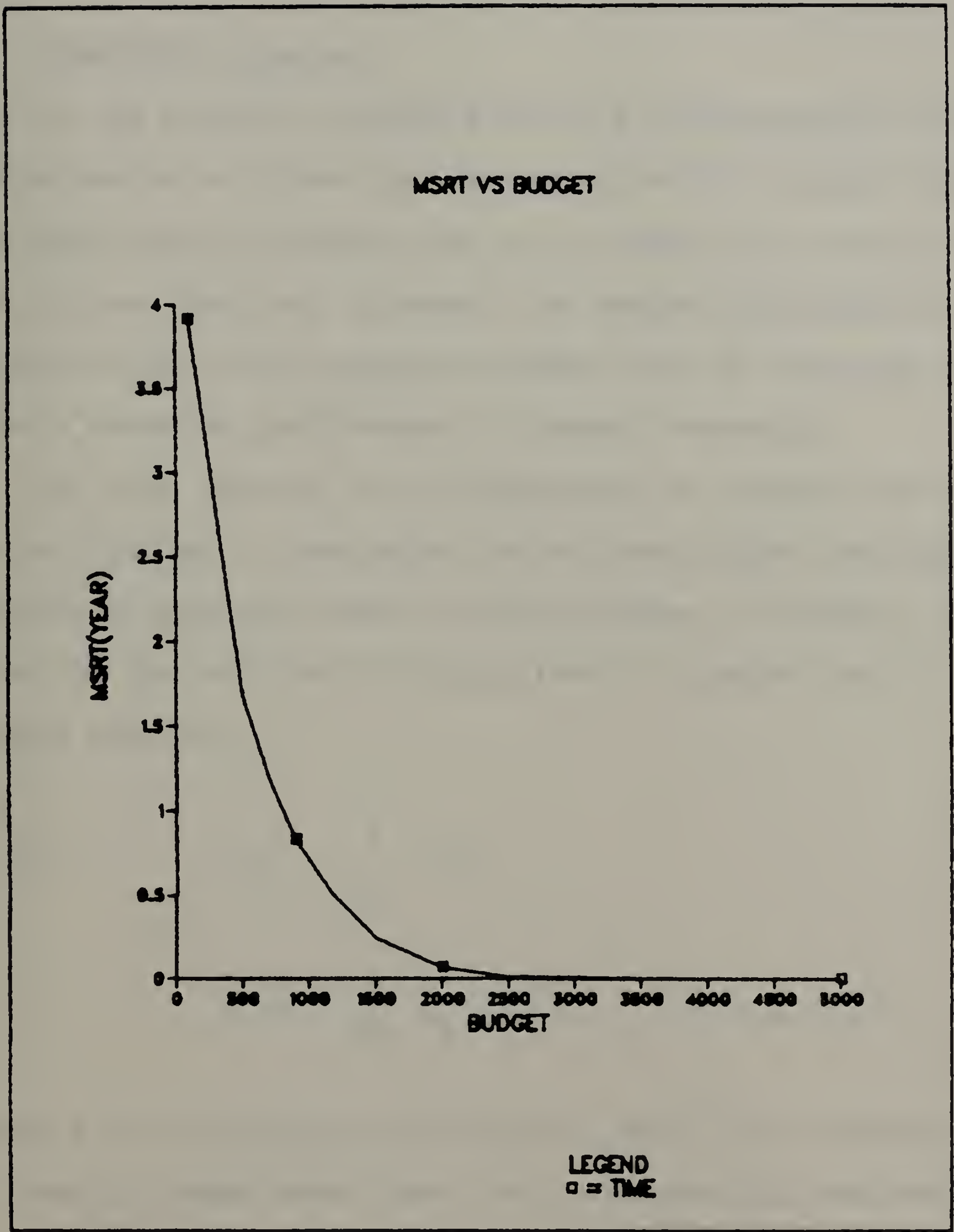

Figure 4. MSRT Vs. Budget Curve

The next section determines analytically the minimum amount budget required by solving a companion problem to the problems discussed earlier in this thesis.

B. COMPANION PROBLEMS

In the previous chapters we have concentrated on the optimization of system effectiveness with a budget constraint. For many weapon systems such as air detection radars, missiles and nuclear delivery systems, the system performance is so important that the necessary budget will be provided to attain whatever performance is deemed necessary.

For such systems it is reasonable to restate the optimization problem to determine the minimum budget required to satisfy a specified level of performance. Consider, for example, the availability optimization problem and the companion problem:

$$\text{(D1)} \qquad \min \quad \sum_{i=1}^{n} c_i s_i$$

$$\text{s.t.} \quad \prod_{i=1}^{n} A_i \geq L \qquad i = 1,2,\ldots,n$$

where L is the minimum performance level for a weapon system.

For the MSRT model case the corresponding problem is:

$$\text{(D2)} \qquad \min \quad \sum c_i s_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} MSRT_i(s_i) \leq R \qquad i = 1,2,\ldots,n$$

where R is the maximum allowable cumulative supply response time for the weapon system.

Problems (D1) and (D2) can be solved using the same methods explained in Chapters III and IV. In the above models the budget is determined so that the system requirement for availability or main supply response time can be achieved.

For problem (D2) the total cost is minimized when

$$\sum_{i=1}^{n} MSRT_i(s_i) = R \; .$$

Sometimes a minimum allowable supply response time is required for each item. In such a case multiple constraints could be specified. This is illustrated below:

$$\text{(D3)} \qquad \min \quad \sum_{i=1}^{n} c_i s_i$$

$$\begin{aligned} MSRT_1(s_1) &\leq R_1 \\ MSRT_2(s_2) &\leq R_2 \\ &\vdots \\ MSRT_n(s_n) &\leq R_n \end{aligned}$$

where R_i is the maximum allowable supply response time, $i = 1,2,...,N$.

To solve (D3), find the smallest s_i such that

$$\begin{aligned}
MSRT_1(s_1) &= R_1 \\
MSRT_2(s_2) &= R_2 \\
&\vdots \\
MSRT_n(s_n) &= R_n
\end{aligned}$$

This problem is solved easily using the same procedures which we discussed.

So far we have discussed many different ways to apply the theoretical models to practical use of models for budget determination decisions. There is no unique method which gives us an optimal result. So the user of these models should choose one of possible methods so as to maximize the system performance or minimize the total cost.

VII. CONCLUSIONS

It is concluded that the various measures of effectiveness can be used in the budget constrained multi-item inventory system with stochastic demands. We have examined some of the more reasonable measures like minimization of units short, minimization of time-weighted units short and maximization of system availability. We have also looked at models which incorporate essentiality weights into each of the models.

In order to solve budget allocation stockage problems a feasible, efficient method of effecting line item inventory control is available using an adaptation of Everett's Generalized Lagrangian Multiplier method. Further, the use of a G.L.M. procedure provides valuable information for system managers as to relative effectiveness of additional procurement funds, versus additional transition processing capability. The final value of Lagrangian multipliers can be interpreted as the amount of improvement of the objective function per unit dollar spent.

The models discussed in this thesis are all more likely to stock cheap, high demand items than expensive, low demand items. Such is the nature of budget constrained optimization problems. If a system manager wishes to maintain enough stock for an item having low demand, high cost, his only alternative in our models is to assign a high essentiality

code for the item. The essentiality code has the effect of reducing the ratio C/E as opposed to C. In the solution procedure for each model, the assigned essentiality code directly affects the allocation for the item.

We have shown how the models can be used as a tool to determine the amount of budget. A simple graphical procedure allows a decision maker to determine the minimum budget required to search a specified level of performance. The optimization model is run several times to generate a plot of performance vs. budget. Each point of the curve represents the effectiveness for the optimal allocation of a given budget. A manager can, first, determine an appropriate system performance level and read from the curve the budget required to achieve the effectiveness.

Further analysis to improve these models may be possible. For instance, it would be useful to have an automatic search algorithm for the Lagrangian multipliers for a multiple constrained problem. It may also be possible to relax the assumptions for a constant lead time or a constant mean time to repair. These single period inventory may expand to time-dependent multi-item, multi-echelon, multi-indenture inventory systems.

APPENDIX A

COMPUTER PROGRAM FOR INTERACTIVE SEARCH

```
      DIMENSICN KDEM(5,100),TMLD(100),IDUES(100),IONHD(100),
     1PRICE(1C0),SZRQN(100),INVNEW(100),PI(100),ZWT(100),TMSVD(
     2100),QUAL(100),KFREQ(5,100)
      COMMON CEMFCT(100), DEVFCT(100),IPOS(100)
      LOGICAL QUAL
      N = 100
      NORDER = 0
      CSTBUD = 0.
      DATA  IC1/'YES'/,IC2/ 'NO'/
      FLAG = 0.
      WRITE(6,137)
  137 FORMAT('   READ IN INITIAL VALUES.')
      READ(3,10) N,(ZWT(I),I=1,4),SHADB,SHADW,TMISS,TMNIS,J,R,WRKLD,
     1BUDGET
      DO 731 L=1,N
  731 READ(4,20) (KDEM(I,L),I=1,5),DEMFCT(L),PRICE(L),(KFREQ(I,L),I=1,5)
     1,DEVFCT(L),TMLD(L),IDUES(L),IONHD(L)
 10   FORMAT(I4, 8F5.2,3F6.2,F10.2)
 20   FORMAT(5I3,1X,2F5.2,1X,5I2,F6.2,5X,F2.1,7X,I2,I3)
 C1   FORMAT(5I3,1X,2F6.2,1X,5I2,F7.2,5X,F3.1,7X,I2,I3)
      DO 22 J=1,N
      DEM= 0.
      DNUM=0.
      DO 23 K=1,5
      DDEM=KDEM(K,J)
      DFREQ=KFREQ(K,J)
      DNUM=DNUM+DDEM
   23 DEM=DEM+DFREQ
      SZRQN(J)=1.
      IF(DEM.LE.0.) GO TO 967
      SZRQN(J)=DNUM/DEM
      IF(SZRQN(J).LE.0.) SZRQN(J)=1.
  967 IPOS(J)= IDUES(J)+IONHD(J)
      TMSVD(J)=AMAX1(TMLD(J)+TMNIS-TMISS,0.)
   22 PI(J)=ZWT(1)*PRICE(J)+ZWT(2)*U+ZWT(3)*R/SZRQN(J)+ZWT(4)*TMSVD(J)
      ZSTARM = 0.
       ZS = 0.
      ZR = 0.
      ZU = 0.
      ZT = 0.
      ZMAXS = 0.
      ZSTAR = 0.
      ZMAXT = 0.
      ZMAXU = 0.
      FLAG=0.
      ZMAXR = 0.
  313 DO 100 J=1,N
      QTY1 = DEMFCT(J)
```

```
      IQTY2 = IPOS(J)
      ZMAXS = ZMAXS + PRICE(J)*QTY1
      VALDSC=DSCVAL(IQTY2,J)
      ZS=VALDSC*PRICE(J)+ZS
      ZU=ZU+U*VALDSC
      ZMAXU=ZMAXU+U*QTY1
      ZMAXR=ZMAXR + R*QTY1/SZRQN(J)
      ZR = ZR + R*VALDSC/SZRQN(J)
      ZMAXT=ZMAXT + QTY1*TMSVD(J)
      ZT = ZT + VALDSC*TMSVD(J)
      ZSTAR = ZSTAR + VALDSC*PI(J)
  100 ZSTARM = ZSTARM +(PI(J))*QTY1
      WRITE(6,40)     (ZWT(I),I=1,4),ZSTAR,ZSTARM,ZS,ZU,ZR,ZT,ZMAXS,ZMAX
     1U,ZMAXR,ZMAXT
   40 FORMAT(7X,'VALUE OF OBJECTIVE FUNCTION EVALUATED AT LEGACY FOR CHO
     1ICE OF WEIGHTS',//,14X,F6.4,2X,'FOR SALES',/,14X,F6.4,2X,'FOR UNIT
     2S ISSUED',/,14X,F6.4,2X,'FOR REQNS ISSUED',/,14X,F6.4,2X,'FOR CUST
     3OMER TIME SAVED',//,6X,'IS',2X,F10.2,'.',//,6X,'IF INFINITE RESOUR
     4CES WERE AVAILABLE THIS VALUE MIGHT BE INCREASED TO',F11.2,'.',//,
     56X,'VALUES OF LEGACY AND GREATEST POSSIBLE VALUE OF THE PURE OBJEC
     6TIVE FUNCTIONS ARE',//,10X,'SALES',16X,'UNITS',16X,'REQNS',15X,'TI
     7ME',/4F20.2,/4F20.2)
      ZSTAR = 0.
      ZS = 0.
      ZU = 0.
      ZT = 0.
      ZR = 0.
  312 DO 200 J=1,N
      ITRIAL = IPOS(J) + 1
      RUSS= SHADB * PRICE(J)/ PI(J)
      D= DEMFCT(J)
      V = DEVFCT(J)
      BILL = ANDTRY(RUSS)
      DEB = BILL*DEVFCT(J) + DEMFCT(J)
      APOS = IPOS(J)
      IF(DEB.LT.APOS) DEB=APOS
      MIKE = DEB + .5
      ITRIAL = MIKE
  201 VALDSC = DSCVAL(ITRIAL,J)
      IBUYQ = ITRIAL - IPOS(J)
      BUYQ = IBUYQ
      TSTBY = PI(J)* VALDSC - (SHADB * PRICE(J) * BUYQ) - SHADW
      ICE = IPOS(J)
      ALICE = DSCVAL(ICE,J)
      TSTNBY = PI(J)* ALICE
      IF(TSTBY.GT.TSTNBY) GO TO 204
      WINNER = ALICE
      INVNEW(J) = IPOS(J)
```

```
 199 ZSTAR= ZSTAR + PI(J)*WINNER                                           INV00970
     ZS = ZS + WINNER*PRICE(J)                                             INV00980
     ZU = ZU + WINNER * U                                                  INV00990
     ZR = ZR + WINNER * R / SZRQN(J)                                       INV01000
     ZT = ZT + WINNER * TMSVD(J)                                           INV01010
     GO TO 200                                                             INV01020
 204 WINNER = VALDSC                                                       INV01030
     INVNEW(J)=ITRIAL                                                      INV01040
     NORDER = NORDER + 1                                                   INV01050
     CSTBUD = CSTBUD +  BUYQ * PRICE(J)                                    INV01060
     GO TO 199                                                             INV01070
 200 CONTINUE                                                              INV01080
     UBOUND = ZSTAR - (CSTBUD - BUDGET)*SHADB -(NORDER- WRKLD)*SHADW       INV01090
     WRITE(6,300) BUDGET, WRKLD                                            INV01100
 300 FORMAT(10X,'BUDGET =',F10.2,10X,'WORKLOAD LIMIT IS',F5.0)             INV01110
 301 WRITE(6,302) CSTBUD, NORDER, UBOUND                                   INV01120
 302 FORMAT(5X,'COST IS',F10.2, 5X,'NORDER =',I4,5X,'UBOUND =',F10.2)      INV01130
     WRITE(6,329) ZSTAR, ZS, ZU, ZR, ZT                                    INV01140
 329 FORMAT(5X,'ZSTAR = ', F8.2,'ZS = ',F8.2,'     ZU =',F8.2,'    ZR      INV01150
    1=',F8.2,'ZT=',F8.2)                                                   INV01160
     WRITE(6,303)                                                          INV01170
 303 FORMAT(3X,'    TRY AGAIN?')                                           INV01180
     READ(5,304) IANS                                                      INV01190
 304 FORMAT(A3)                                                            INV01200
     IF(IANS.EQ.IC2) GO TO 400                                             INV01210
     ZSTAR = 0.                                                            INV01220
     ZU = 0.                                                               INV01230
     CSTBUD = 0.                                                           INV01240
     ZS = 0.                                                               INV01250
     NORDER = 0.                                                           INV01260
     ZR = 0.                                                               INV01270
     ZT = 0.                                                               INV01280
     ZMAXS = 0.                                                            INV01290
     ZMAXT = 0.                                                            INV01300
     ZMAXR = 0.                                                            INV01310
     ZMAXU = 0.                                                            INV01320
     FLAG = 0.                                                             INV01330
     ZSTARM = 0.                                                           INV01340
     WRITE(6,305)                                                          INV01350
 305 FORMAT(3X,'    SAME WEIGHTS?')                                        INV01360
     READ(5,304) IANS                                                      INV01370
     IF(IANS.EQ.IC1) GO TO 350                                             INV01380
     FLAG=1.0                                                              INV01390
 310 WRITE(6,307)                                                          INV01400
 307 FORMAT(3X,'   INPUT ZS,ZU,ZR,ZT VIA 4F5.3')                           INV01410
     READ(5,306) ZWT(1),ZWT(2),ZWT(3),ZWT(4)                               INV01420
 306 FORMAT(4F5.3)                                                         INV01430
     WRITE(6,308)(ZWT(I),I=1,4)                                            INV01440
```

```
  308 FORMAT(3X,' WEIGHTS ARE',3X,4(F6.3,2X))
      WRITE(6,309)
  309 FORMAT(3X,'  CORRECT?')
      READ(5,304) IANS
      IF(IANS.EQ.IC2) GO TO 310
      DO  800 J=1,N
800   PI(J)=ZWT(1)*PRICE(J)+ZWT(2)*U+ZWT(3)*R/SZRQN(J)+ZWT(4)*TMSVD(J)
350   WRITE(6,363) SHADB,SHADW
  363 FORMAT(3X,'SHADB =',F10.3,5X,'SHADW =',F10.3,/,'    SAME MULTIPLIER
     1S?')
      READ(5,304) IANS
      IF(IANS.EQ.IC1) GO TO 312
      WRITE(6,315)
  315 FORMAT(3X,' INPUT SHADB AND SHADW VIA 2F10.3')
  316 FORMAT(2F10.3)
      READ(5,316) SHADB,SHADW
      GO TO 312
  400 WRITE(6,401)
  401 FORMAT(3X,' LOOK AT LINE ITEMS?')
      READ(5,304) IANS
      IF(IANS.EQ.IC2) GO TO 500
      WRITE(6,981)
981   FORMAT('    HOW MANY?',/,' XXX')
      READ(5,982)NUMI
982   FORMAT(I3)
      WRITE(6,403)
403   FORMAT('  ITEM  LEGACY  NEW QTY')
      DO 402 J=1,NUMI
      IF(IPOS(J).EQ.INVNEW(J) ) GO TO 402
      WRITE(6,404) J, IPOS(J), INVNEW(J)
404   FORMAT(2X,I4,3X,I5,3X,I5)
402   CONTINUE
      WRITE(6,403)
      DO 405 J=1,N
      WRITE(6,404) J, IPOS(J),INVNEW(J)
405   CONTINUE
  500 STOP
      END
      FUNCTION DSCVAL(I,J)
      COMMON DEMFCT(100), DEVFCT(100), IPOS(100)
      K=1
      DSCVAL = 0.
      IF(I.LE.0) GO TO 182
      D = DEMFCT(J)
      V = DEVFCT(J)
180   DSCVAL = DSCVAL + DLTAZJ(K,D,V)
      IF(K.GE.I) GO TO 182
      K=K+1
```

```
      GO TO 180
182   RETURN
      END
      FUNCTION DLTAZJ(I,D,V)
      COMMON DEMFCT(100),DEVFCT(100),IPOS(100)
      SIGMA = 1.25*V
      AMIN = D* .5
      IF(SIGMA.LT.AMIN) SIGMA = AMIN
      IF(SIGMA.LT.1.) SIGMA = 1.
      AI = I
      X =(AI - D)/SIGMA
      AX = ABS(X)
      T = 1.0/(1.0 + .2316419*AX)
      D = 0.3989423 * EXP(-X*X/2.0)
      DLTAZJ = 1.0 - D*T*((((1.330274*T - 1.821256)*T + 1.781478)*T
     1 - 0.3565638)*T + 0.3193815)
      DLTAZJ = 1.0 - DLTAZJ
      IF(X) 1,2,2
1     DLTAZJ= 1.0 - DLTAZJ
2     RETURN
      END
      FUNCTION ANCTRY(APROB)
      IF(APROB.GT..995)APROB=.995
      IF(APROB.LT..005)APROB=.005
 7    D = APROB
      IF(D - 0.5) 9,9,8
 8    D = 1.0 - D
 9    T2 = ALOG(1.0 /(D*D))
      T = SQRT (T2)
      X = T - (2.515517+0.802853*T+0.010328*T2)/(1.0+1.432788*
     1T+0.189269*T2 + 0.001308*T*T2)
      IF(APROB - .5) 10,10,11
 10   X = -X
 11   D = 0.3989423*EXP(-X*X/2.0)
      X = -X
      ANDTRY = X
 12   RETURN
      END
```

APPENDIX B

COMPUTER PROGRAM FOR AUTOMATING SEARCH

```
C      VARIABLE DEFINITION
C
C      TMLD     : LEAD TIME  (100)
C      TMEAN    : MEAN DEMAND (100)
C      TMLDM    : LEAD TIME DEMAND (100)
C      PRICE    : PRICE(100)
C      DELZ     : (Z(1)-Z(0))/PRICE(I)   (100)
C      S        : THE NUMBER OF ITEM REQUIRED (100)
C      PBAR     : P(S) COMPLEMENTRY CUMULATIVE DISTRIBUTION  (100)
C      ZSTAR    : THE VALUE OF OBJECTIVE FUNTION
C      CSTBUD   : USED BUDGET
C      BUDGET   : GIVEN BUDGET
C      UB       : UPPER BOUND
C      CUB      : CURRENT UPPER BOUND
C      THETA    : LAGLANGIAN MULTIPLIER
C      MTBF     : MEAN TIME BETWEEN FAILURE
C      MSRT     : MEAN SUPPLY RESPONSE TIME
C      MTTR     : MEAN TIME TO REPAIR
C      ESS      : ESSENTIALITY CODE
C
C
C    ===================================================================
      DIMENSICN DELZ(100),S(100),PBAR(100),MTTR(100),ESS(100),AAV(100),
     /TMLD(100),TMEAN(100),TMLDM(100),PRICE(100),MTBF(100),MSRT(100)
      REAL*4 DELZ,S,PBAR,ZSTAR,CSTBUD,UB,CUB,THETA,THETAM,THETAL,
     1Z1,SS,SNN,SN,SSS,OLD,X1,X2,X3,X4,X5,X6,X7,X8,CCDF,PO,ZS1,ZS2,
     1Z2,Z3,ZO,MTBF,MTTR,ESS,MSRT,AAV
      INTEGER*4 I,J,K,N,KK
C
C
C    ===================================================================
C     INPUT DATA
C    ===================================================================
      N=10
      BUDGET=19224.0
      DO 11 I=1,N
         READ(5,12) TMEAN(I),TMLD(I),PRICE(I),MTTR(I),ESS(I)
         MTBF(I)=1.0/TMEAN(I)
   11 CONTINUE
   12 FORMAT(5F10.5)
C    ==================================================================
C     FIND Z(1)-Z(0) ARRAY
C    ==================================================================
      DO 13 I=1,N
       TMLDM(I)=TMEAN(I)*TMLD(I)
      CALL SUB (TMLDM(I),1.0,CCDF,CDF,PO)
         X1=CCDF
         X3=PO
```

```
      CALL SUB (TMLDM(I),0.0,CCDF,CDF,PO)
         X2=CCDF
      Z1=(TMLC(I)-(1.0/TMLDM(I)))*X2
      Z1=Z1+(1.0/TMLDM(I))*X3
      DELZ(I)=Z1*ESS(I)/PRICE(I)
   13 CONTINUE
C   =================================================================
C
C     FIND UPPER VALUE OF THETA
C
C    ================================================================
      THETAM=DELZ(1)
      DO 14 K=2,N
         IF(THETAM.GT.DELZ(K)) GO TO 14
            THETAM=DELZ(K)
   14 CONTINUE
C     =================================================================
C
      THETAL=0.0
      THETA=THETAM/2.0
      OLD=20.
      WRITE(6,906) THETAM
  906 FORMAT(1X,' THE MAXIMUM VALUE OF THETA = ',F10.2)
C
C    ================================================================
C     FIND S(1),S(2), , , , S(N)
C   =================================================================
  800 CONTINUE
      SUM=0.0
      DO 901 J=1,N
         SSS=1.0
  903    S1=SSS-1.0
         S2=SSS+1.0
         S3=SSS+2.0
      CALL SUB (TMLDM(J),S1,CCDF,CDF,PO)
         X1=CCDF
      CALL SUB (TMLDM(J),SSS,CCDF,CDF,PO)
         X2=CCDF
         X5=PO
          ZS1=(TMLC(J)-(SSS/TMLDM(J)))*X1
          ZS1=(ZS1+(SSS/TMLDM(J))*X5)/PRICE(J)
          ZS1=ESS(J)*ZS1
      IF(X2.LE.0.0001) GO TO 902
      IF( ZS1.LE.(THETA)) GO TO 902
      SSS=SSS+1.0
       GO TO 903
  902  S(J)=SSS-1.0
       SUM=SUM+S(J)*PRICE(J)
```

```
  901  CONTINUE
       WRITE(6,905) (S(I),I=1,10),THETA,SUM
   905 FORMAT(1X,' S=',10F5.1,' THETA= ',F9.5,'CST=',F9.1)
C
C      ==================================================================
C          COMPUTE Z(S) BY USING S(1),S(2),,,,,,S(N)
C      ==================================================================
       UB=99999.0
       ZSTAR=0.0
       ZSTARL=0.0
       CSTBUD=0.0
       SLT=0.0
       DO 15  J=1,N
          SLT=SLT+TMLDM(J)
C      CALL SUB (TMEAN(J),S(J)-1.0,CCDF,CDF,PO)
C         X5=CCDF
       CALL SUB (TMLDM(J),S(J),CCDF,CDF,PO)
          X6=CCDF
       CALL SUB (TMLDM(J),S(J)+1.0,CCDF,CDF,PO)
          X7=CCDF
       CALL SUB (TMLDM(J),S(J)+2.0,CCDF,CDF,PO)
          X8=CCDF
          PBAR(J)=X6
          Z1=TMLDM(J)*X6
          Z2=2.0*S(J)*X7
          Z3=S(J)*(S(J)+1.0)*X8/TMLDM(J)
          Z0=0.5*TMLD(J)*(Z1-Z2+Z3)
          MSRT(J)=ZC/TMLDM(J)
          AAV(J)=MTBF(J)/(MTBF(J)+MTTR(J)+MSRT(J))
          ZSTAR=ZSTAR+Z0*ESS(J)
           CSTBUD=CSTBUD+S(J)*PRICE(J)
   15  CONTINUE
       ZSTAR=ZSTAR/SLT
       CUB=ZSTAR-THETA*(CSTBUD-BUDGET)
C     ===================================================================
C      SEARCH OPTIMAL THETA
C     ===================================================================
C      IF ((CUB-ZSTAR)/CUB.LE.0.00001.OR.(CSTBUD-BUDGET).LT.1.0) GO TO
C    / 99
       IF (ABS(THETA-OLD).LE.0.00001.AND.(CSTBUD.LE.BUDGET)) GO TO 99
C      IF(ABS((CUB-UB)/CUB).LE. 0.001) GO TO 99
C      IF(((BUDGET-CSTBUD).LE.1.).AND.(CSTBUD.GT.BUDGET)) GO TO 99
       IF(CUB.LE.UB) UB=CUB
       IF(CSTBUD.GT.BUDGET) GO TO 21
          THETAM=THETA
          THETA=(THETAL+THETAM)/2.0
          OLD=THETAM
          GO TC 800
```

```
  21      THETAL=THETA
          OLD=THETA
          THETA=(THETAL+THETAM)/2.0
          GO TO 80C
C
C
C     ==============================================================
C      OUTPUT PRINT
C     ==============================================================
C
 99    WRITE(6,100)
       WRITE(6,101) THETA, ZSTAR,CUB,BUDGET,CSTBUD
       WRITE(6,102)
       DO 200 I=1,N
        WRITE(6,103) I,PRICE(I),TMEAN(I),S(I),PBAR(I),AAV(I),MSRT(I)
  200 CONTINUE
  100 FORMAT(1H1,///20X,' ******* THE RESULT OF SEARCH *******'///)
  101 FORMAT(10X,' THETA                = ',F10.4/
     /        10X,' ZSTAR                = ',F10.4/
     /        10X,' CURRENT UPPER BOUND = ',F10.3/
     /        10X,' BUDGET LIMIT         = ',F10.3/
     /        10X,' USED BUDGET          = ',F10.3//)
  102 FORMAT(6X,' NO. ',5X,'  PRICE   ',5X,'M DEMAND ',5X,'  BUY S ',5X,
     /' PBAR (S) ',' AVIL (S) ',5X,' MSRT(S) '//)
  103 FORMAT(6X,I4,7X,F9.2,5X,F9.2,5X,F8.2,5X,F8.3,5X,F8.3,5X,F9.5)
      STOP
      END
      SUBROUTINE SUB(TT,SS,CCDF,CDF,PO)
      REAL*4 TT,SN,SS,PO,CDF,CCDF
      INTEGER*4 I
      SN=0.0
      PO=EXP(-TT)
      CDF=PO
      CCDF=1.0-CDF
C     WRITE(6,10) TT,SN,PO,CDF,CCDF
C  10 FORMAT(1X,' MEAN,SN,PO,CDF,CCDF ',5F9.5)
C     IF(CDF.GE.0.005) CCDF=0.005
      IF(SS.LE.0.C) GO TO 26
      SN=SN+1.0
  22  PO=PO*TT/SN
      CDF=CDF+PO
      CCDF=1.0-CDF
C     WRITE(6,11) TT,SN,PO,CDF,CCDF
C  11 FORMAT(1X,' MEAN,SN,PO,CDF,CCDF, '5F9.5)
      IF(CDF.GT.0.9999) GO TO 26
      IF(SN.GE.SS) GO TO 26
      SN=SN+1.0
      GO TO 22
```

```
26   CONTINUE                                                      TH201930
     RETURN                                                        TH201940
     END                                                           TH201950
```

APPENDIX C

COMPUTER PROGRAM FOR MARGINAL ANALYSIS

```
C
C ====    TEST FOR THE THREE INVENTORY MODELS ======
C
C ==================
C  ** OBJECTIVES **
C  ================
C
C            1. MINIMIZE UNITS SHORT MODEL
C            2. MINIMIZE TWUS        MODEL
C            3. MAXIMIZE PSEUDO AVAILABILITY MODEL
C
C  ** CONSTRAINT **
C            -- BUDGET ----
C
C  ** METHOD **
C            -- MARGINAL ANALYSIS TECHNIQE --
C
C ** FUNCTIONS ***
C            1.PCCDF:  FIND C.C.D.F FOR POISSON DISTRIBUTION
C            2.PDEN :  FIND PROB(X=N) POSSION DISTRIBJTION
C            3.TWS  :  FIND THE OBJECT FUNCTION FOR TWUS MODEL
C            4.UNITS:  FIND THE OBJECT FUNCTION
C            5.DELU :  FIND THE (Z(S-1)-Z(S))*E/C FOR MODEL 1
C            6.DELTWS: FIND THE (Z(S-1)-Z(S))*E/C FOR MODEL 2
C            7.DELAV : FIND THE (Z(S)-Z(S-1))*E/C FOR MODEL 3
C
C ** VARIABLE DEFINITION
C            LAM(I)   : POISSON ANNUAL DEMAND
C            T(I)     : LEAD TIME
C            C(I)     : PRICE FOR AN ITEM
C            ESS(I)   : ESSENTIALITY CODE
C            MTTR(I)  : MEAN TIME TO REPAIR
C            MSRT(I)  : MEAN SUPPLY RESPONSE TIME
C            AAV(I)   : AVAILABILITY
C            US (I)   : UNIT SHORT
C            S(I)     : STOCK VECTOR
C            RATIO(I): Z(S)-Z(S+1)
C
C
      REAL LAM(100),T(100),C(100),ESS(100),MTTR(100),MSRT(100),
     /AAV(100),US(100),S(100),RATIO(100),
     /B,BR,RR,OBJ1,OBJ2,OBT3,LTM,MTBF,X
      INTEGER N,I,K J,KK
C
C     INPUT DATA
C
      WRITE(6,9)
      N=10
```

```
      DO 1 I=1,N
         READ(5,12) LAM(I),T(I),C(I),MTTR(I),ESS(I)
         WRITE(6,13) I,LAM(I),T(I),C(I),ESS(I),MTTR(I)
    1 CONTINUE
    9 FORMAT(1H1,10X,'    NO',5X,'LAMDA',5X,'LEAD T',5X,'PRICE ',5X,
     /' ESS.',5X,' MTTR   '//)
   12 FORMAT(5F10.4)
   13 FORMAT(10X,I5,5X,F5.1,5X,F5.1,5X,F6.1,5X,F5.1,5X,F10.4)
      B=1170.
C-----------------------------------------------------------------------
C
C     INITIALIZE
C
      DO 15 I=1,N
         S(I)=0.0
C        RATIC(I)=DELU(LAM(I),T(I),1.0,C(I),ESS(I))
C        RATIO(I)=DELTWS(LAM(I),T(I),1.0,C(I),ESS(I))
         RATIO(I)=DELAV(LAM(I),T(I),1.0,C(I),ESS(I),MTTR(I))
   15 CONTINUE
C
C-----------------------------------------------------------------------
C
C    ALLOCATE ITEMS WITH GIVEN BUDGET
C
C
      BR=B
   19 KK=0
      RR=-1.
      DO 16 K=1,N
         IF(C(K).GT.BR) GO TO 16
         IF(RATIO(K).LE.RR) GO TO 17
         RR=RATIO(K)
         KK=K
   17    CONTINUE
   16 CONTINUE
      IF(KK.EQ.0) GO TO 18
C
C     UPDATE RATIC(K),S(K),BR
C
      S(KK)=S(KK)+1.0
C     RATIO(KK)=DELU(LAM(KK),T(KK),S(KK),C(KK),ESS(KK))
C     RATIO(KK)=DELTWS(LAM(KK),T(KK),S(KK),C(KK),ESS(KK))
      RATIO(KK)=DELAV(LAM(KK),T(KK),S(KK),C(KK),ESS(KK),MTTR(KK))
      BR=BR-C(KK)
      THETA=RR
      GO TO 19
C
C-----------------------------------------------------------------------
```

```
C
C     CALCULATE OBJECTIVE FUNTIONS WITH S VECTORS OBTAINED FORM THE
C        ABOVE STEP
C
   18 OBJ1=0.0
      OBJ2=0.C
      OBJ3=1.0
      SUMLTM=0.0
      DO 20 J=1,N
         LTM=LAM(J)*T(J)
         X=S(J)
         MTBF=1.0/LAM(J)
         SUMLTM=SLMLTM+LTM
         US(J)=UNITS(LAM(J),T(J),X)
         MSRT(J)=TWS(LAM(J),T(J),X)/LTM
         AAV(J)=MTBF/(MTBF+MTTR(J)+MSRT(J))
         OBJ1=OBJ1+ESS(J)*US(J)
         OBJ2=OBJ2+ESS(J)*MSRT(J)*LTM
         OBJ3=OBJ3*AAV(J)
   20    CONTINUE
         OBJ1=OBJ1/SUMLTM
         OBJ2=OBJ2/SUMLTM
C
C-----------------------------------------------------------------------
C
C
C     OUTPUT FCR THIS TEST
C
C
      WRITE(6,25)
      DO 26 I=1,N
         WRITE(6,27) I,S(I),US(I),MSRT(I),AAV(I)
   26 CONTINUE
   25 FORMAT(///1X,'  ==== ALLOCATION OF ITEMS  ======'///
     /5X,'  I  ',10X,' S(I) ',5X,'  US(I)  ',5X,'MSRT(I)',5X,' AVIL ')
   27 FORMAT(5X,I3,12X,F6.1,5X,F9.4,3X,F9.4,5X,F9.5)
      WRITE(6,30) B,BR,THETA,OBJ1,OBJ2,OBJ3
   30 FORMAT(//10X,'BUDGET LIMIT =',F10.2/
     /         10X,'BUDGET LEFT =',F10.2/
     /         10X,'THETA        =',F10.6/
     /         10X,'OBJECT VALUE OF UNITS SHORT MODEL  =',F10.6/
     /         10X,'OBJECT VALUE OF TWUS        MCDEL  =',F10.6/
     /         10X,'OBJECT VALUE OF AVAIL.      MODEL  =',F10.6//)
      STOP
      END
C    POISSON COMPLMENTRY CUMULATIVE DISTRIBUTION FUNTIJN
C
      REAL FUNCTICN PCCDF(AL,TT,SS)
      REAL AL,TT,SS,PG,CDF,SN,AA
```

```
      AA=AL*TT
      SN=0.0
      PCCDF=1.C
      IF(SS.LE.0.0) GO TO 26
      PO=EXP(-AA)
      CDF=PO
      PCCDF=1.-CDF
      IF(SS.EQ.1.0) GO TO 26
C     IF(CDF.GT.0.9999) PCCDF=0.0001
C     IF((SS-1.0).EQ.0.0) GO TO 26
   22 SN=SN+1.C
      PO=PO*AA/SN
      CDF=CDF+PO
      PCCDF=1.C-CDF
      IF(PCCDF.LE.0.00001) GO TO 26
      IF(SN.GE.(SS-1.0)) GO TO 26
C     IF(CDF.GT.0.99999) PCCDF=0.0001
C     SN=SN+1.0
      GO TC 22
   26 CONTINUE
      RETURN
      END
C
C     POISSON MASS FUNTION
C
      REAL FUNCTICN PDEN (AL,TT,SS)
      REAL AL,TT,SS,AA
      SN=1.0
      AA=AL*TT
      PDEN=EXP(-AA)
      IF(SS.LE.0.0) GO TO 27
   28 PDEN=PDEN*AA/SN
      IF(SS.EQ.SN) GO TO 27
      SN=SN+1.0
      GO TC 28
   27 CONTINUE
      RETURN
      END
C
C     TIME WAITED UNITS SHORT OBJECTIVE FUNTICN
C
      REAL FUNCTICN TWS(AL,TT,S1)
      REAL AL,TT,S1,Z1,Z2,Z3,AA
      AA=AL*TT
      Z1=AA*PCCDF(AL,TT,S1)
      Z2=2.0*S1*PCCDF(AL,TT,S1+1.0)
      Z3=S1*(S1+1.0)*PCCDF(AL,TT,S1+2.0)/AA
      TWS=0.5*TT*(Z1-Z2+Z3)
```

```
      RETURN
      END
C
C     CALCULATE THE OBJECTIVE FUNTION FOR UNIT SHORT MODEL
C
      REAL FUNCTICN UNITS(AL,TT,S2)
      REAL AL,TT,S2
      UNITS=AL*TT*PCCDF(AL,TT,S2-1.0)-S2*PCCDF(AL,TT,S2)
C     UNITS=AL*TT*PDEN(AL,TT,S2)+(AL*TT-S2)*PCCDF(AL,TT,S2+1.)
      RETURN
      END
C
C    THIS FUNCTION 'DELU' CALCULATES (E/C(Z(S-1)-Z(S)) FOR UNITS
C    SHORT MODEL
C
      REAL FUNCTICN DELU(AL,TT,S2,CC,E)
      REAL AL,S2,TT,CC,E
      DELU=PCCDF(AL,TT,S2)
C     DEL1=UNITS(AL,TT,S2-1.0)
C     DEL2=UNITS(AL,TT,S2)
C     DELU=DEL1-DEL2
      DELU=(E/CC)*DELU
      RETURN
      END
C
C   THIS FUNTION COMPUTES  THE MARGINAL EFFECTAS INCREASE ONE MORE
C    ITEMS FOR TWLS MODEL CASE

      REAL FUNCTICN DELTWS(AL,TT,SS,CC,E)
      REAL AL,TT,SS,CC,E,DEL1,DEL2,DEL3
      DEL1=TWS(AL,TT,SS)
      DEL2=TWS(AL,TT,SS-1.0)
      DELTWS=(E/CC)*(DEL2-DEL1)
C     DEL3=(TT-SS/AL)*PCCDF(AL,TT,SS)-(SS/AL)*PDEN(AL,TT,SS)
C     DELTWS=(E/CC)*DEL3
C
C
      RETURN
      END
C
C    THIS FUNCTION COMPUTE THE MARGINAL EFFECT  AS INCREASE ONE MORE ITE
C         ( PSEUDC AVAILABILITY MODEL CASE)
C
      REAL FUNCTICN DELAV(AL,TT,S4,CC,E,MTR)
      REAL AL,TT,CC,E,MTR,S4,A1,A2,MST1,MST2,MTF
      MST1=TWS(AL,TT,S4)/(AL*TT)
      MST2=TWS(AL,TT,S4-1.0)/(AL*TT)
      MTF=1.0/AL
```

```
A1=MTF/(MTF+MTR+MST1)
A2=MTF/(MTF+MTR+MST2)
DELAV=(ALOG(A1)-ALOG(A2))*E/CC
RETURN
END
```

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